The Uniqueness of the Adjoint Operation. II

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ABSTRACT

Let H be a complex, finite-dimensional Hilbert space, and let $\mathrm{L}(H)$ denote the set of linear transformations mapping H into itself. For certain interesting subsets $A(H)$ of $\mathrm{L}(H)$ [nonsingular transformations and $\mathrm{L}(H)$ are examples], the functions $h: A(H) \to \mathrm{L}(H)$ which have the properties $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ are characterized.

1. INTRODUCTION AND MAIN RESULTS

Let H be a complex, finite-dimensional Hilbert space with inner product $(\cdot,\cdot)$. Let $\mathrm{L}(H)$ denote the set of linear transformations mapping H into itself, let $\mathrm{GL}(H) \subset \mathrm{L}(H)$ denote the set of nonsingular transformations, let $\mathrm{U}(H) \subset \mathrm{GL}(H)$ denote the set of unitary transformations (i.e. $UU^* = I = U^*U$), and let $\mathrm{SU}(H) \subset \mathrm{U}(H)$ denote the set of unitary transformations with determinant one. For the moment, let $A(H)$ denote one of these sets, and let $h: A(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in A(H)$. Recall that if $T \in \mathrm{L}(H)$, then $T \geqslant 0$ means that $(Tx, x) \geqslant 0$ for all $x \in H$. What can be said about h? Well, $h(S) = S^*$ always works, and $h(S) = S^{-1}$ works for appropriate choices of $A(H)$. The purpose of this paper is to characterize the functions h for any of the allowable $A(H)$'s. The results are as follows:

THEOREM 2.1. *Let h: $\mathrm{SU}(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{SU}(H)$. If $h(I) \neq 0$, then $h(S) = S^*$ for all $S \in \mathrm{SU}(H)$.*

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Let $\mathbb{T}$ denote the group under multiplication of complex numbers of modulus 1, and let $\mathbb{R}^+$ denote the group under multiplication of strictly positive real numbers.

THEOREM 2.2. *Let $h: \mathrm{U}(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{U}(H)$. If $h(I) \neq 0$, then there exists a homomorphism $\tau: \mathbb{T} \to \mathbb{R}^+$ (τ need not be continuous) such that $h(S) = \tau(\det(S))S^*$ for all $S \in \mathrm{U}(H)$.*

THEOREM 3.1. *Let $\dim H \geqslant 2$, and let $\mathrm{A}(H) \subseteq \mathrm{L}(H)$ be a multiplicative semigroup which contains $\mathrm{GL}(H)$ and a nonzero singular transformation. If $h: \mathrm{A}(H) \to \mathrm{L}(H)$ is a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{A}(H)$, then either $h(T) = 0$ for all singular $T \in \mathrm{A}(H)$ or $h(S) = S^*$ for all $S \in \mathrm{A}(H)$.*

THEOREM 4.1. *Let $h: \mathrm{GL}(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{GL}(H)$. If $h(I) \neq 0$, then there exists a multiplicative homomorphism $\sigma: \mathbb{C} - \{0\} \to \mathbb{R}^+$ (σ need not be continuous) such that either $h(S) = \sigma(\det(S))S^*$ for all $S \in \mathrm{GL}(H)$ or $h(S) = \sigma(\det(S))S^{-1}$ for all $S \in \mathrm{GL}(H)$.*

The paper concludes with an application of Theorem 4.1.

As the title of this paper suggests, there is another paper concerning this topic. This other paper [2] deals with infinite-dimensional, complex Hilbert spaces. The main result of that paper is a version of Theorem 3.1 which doesn't mention singular transformations.

Any results that appear in this paper without a reference can probably be found in [1].

2. UNITARY TRANSFORMATIONS

LEMMA 2.1. *Let $h: \mathrm{SU}(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{SU}(H)$. If $h(I) \neq 0$ then $h(I) = I$.*

Proof. Since $h(I)I \geqslant 0$ and $h(I \cdot I) = h(I)h(I)$, it follows that $h(I) = I$ if $\dim H = 1$. Let $\dim H \geqslant 2$. If $h(I)x \neq x$ for all nonzero vectors x, then $h(I)x = 0$ for all vectors x. This is true because $h(I)(h(I)y) = h(I)y$ for all $y \in H$. Thus, there exists a nonzero vector x_1 such that $h(I)x_1 = x_1$.

If $h(I)x = x$ for all $x \in H$, then the proof is complete. Suppose there exists a vector y_0 such that $h(I)y_0 \neq y_0$. Let $y_1 = y_0 - h(I)y_0$. Then (x_1, y_1)

$=(h(I)x_1, y_1)=(x_1, h(I)y_1)$ [as the first line of the proof shows $h(I)$ is self-adjoint] $=(x_1,0)=0$. Without loss of generality, it may be assumed that both x_1 and y_1 have length one. Let U be the transformation defined by $Ux_1=-y_1$, $Uy_1=x_1$ and $Uz=z$ for all vectors z orthogonal to both x_1 and y_1. Then $U \in \mathrm{SU}(H)$ (consider the matrix of U with respect to an orthonormal basis containing x_1 and y_1). $h(U)y_1=0$, as $h(U)y_1=h(I\cdot U)y_1=h(U)h(I)y_1$. Thus, $h(U)Ux_1=0$. This and the fact that $h(U)U$ is self-adjoint show that $U^*[h(U)]^*x_1=0$. Since U^* is nonsingular, it follows that $[h(U)]^*x_1=0$. Then $1=(x_1, x_1)=(h(I)x_1, x_1)=(h(U^4)x_1, x_1)=(h(U)h(U^3)x_1, x_1)=(h(U^3)x_1,[h(U)]^*x_1)=0$. ∎

LEMMA 2.2 [2, Lemma 3.2]. *Let $h:\mathrm{U}(H)\to \mathrm{L}(H)$ be a function such that $h(ST)=h(T)h(S)$ and $h(S)S\geqslant 0$ for all $S,T\in \mathrm{U}(H)$. If $h(I)\neq 0$, then $h(J)=J$ for all symmetries $J\in \mathrm{U}(H)$ (i.e. $J=J^*=J^{-1}$).*

Proof (included to keep the paper self-contained). $h(I)=I$, as h satisfies the hypotheses of Lemma 2.1. $h(-I)=-I$ because $h(-I)(-I)\geqslant 0$, $I=h((-I)^2)=(h(-I))^2$, and the positive square root of a positive transformation is unique. This shows that $h(J)=J$ for all symmetries J when the dimension of H is one. Let $\dim H\geqslant 2$. If $J\neq \pm I$ is a symmetry on H, then there exists a nontrivial subspace K of H such that $J=I\oplus -I$ with respect to $K\oplus K^{\perp}$. Let $W=I\oplus iI$ with respect to $K\oplus K^{\perp}$ $[i=(-1)^{1/2}]$. Then $W\in U(H)$ and $W^2=J$. Let

$$h(W)=\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

with respect to $K\oplus K^{\perp}$. Since $h(W)W\geqslant 0$, it follows that

$$\begin{pmatrix} P & iQ \\ R & iS \end{pmatrix}\geqslant 0.$$

Thus, $P\geqslant 0$, $iS\geqslant 0$, and $R=-iQ^*$. This shows that

$$h(W)=\begin{pmatrix} P & Q \\ -iQ^* & S \end{pmatrix} \quad\text{and}\quad h(J)=\begin{pmatrix} P^2-iQQ^* & * \\ * & * \end{pmatrix}.$$

Since $h(J)J\geqslant 0$, it follows that $P^2-iQQ^*\geqslant 0$. Since $P^2\geqslant 0$, it follows that iQQ^* is self-adjoint. Thus, $Q=0$. This shows that

$$h(W)W=\begin{pmatrix} P & 0 \\ 0 & iS \end{pmatrix} \quad\text{and thus}\quad h(W)=\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}.$$

This leads to $I = h(I) = h(W^4)W^4 = [h(W)]^4W^4 = [h(W)W]^4$. This shows that $P^4 = I$ and $(iS)^4 = I$. Since $P \geqslant 0$ and $iS \geqslant 0$, it follows that $P = I$ and $iS = I$. Hence, $h(W) = W^*$ and $h(J) = J$, as $(W^*)^2 = J$. ∎

LEMMA 2.3. *Let $h:\mathrm{U}(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{U}(H)$. If $h(I) \neq 0$ then $h(U) = U^*$ for all $U \in \mathrm{SU}(H)$.*

Proof. If $\dim H = 1$, then this lemma follows from Lemma 2.1. Let $\dim H \geqslant 2$. By Lemma 2.2, it suffices to show that if $U \in \mathrm{SU}(H)$, then there exist symmetries $J_1, J_2, \ldots, J_k$ such that $U = J_1 J_2 \cdots J_k$. From this it follows that $h(U) = h(J_k) \cdots h(J_2)h(J_1) = J_k \cdots J_2 J_1 = U^*$.

Since U is unitary, there exists an orthonormal basis for H such that $U = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with respect to this basis, where the λ_i's are complex numbers of modulus one. Then

$$\begin{aligned}\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = {} & \mathrm{diag}(\lambda_1, \bar{\lambda}_1, 1, 1, \ldots, 1) \\ & \times \mathrm{diag}\left(1, \lambda_1\lambda_2, \overline{\lambda_1\lambda_2}, 1, \ldots, 1\right) \cdots \\ & \times \mathrm{diag}\left(1, \ldots, 1, \lambda_1\lambda_2 \cdots \lambda_{n-1}, \overline{\lambda_1\lambda_2 \cdots \lambda_{n-1}}\right) \\ & \times \mathrm{diag}(1, \ldots, 1, \lambda_1\lambda_2 \cdots \lambda_n).\end{aligned}$$

The last matrix is the identity matrix, because the determinant of U is one. Thus, $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a product of matrices of the form $\mathrm{diag}(1, \ldots, 1, \lambda, \bar{\lambda}, 1, \ldots, 1)$, where the modulus of λ is one. Since

$$\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

a product of symmetries, it follows that $\mathrm{diag}(1, \ldots, 1, \lambda, \bar{\lambda}, 1, \ldots, 1)$ can be written as a product of symmetries. ∎

In the following it will be necessary to take nth roots of complex numbers, where n is the dimension of H. This is a problem, because $(bc)^{1/n}$ may not equal $b^{1/n}c^{1/n}$ for complex numbers b and c; in fact, $\bar{b}^{1/n}$ may not equal $\overline{b^{1/n}}$. Nothing can be done about the first problem, but to remove the second problem, once a number has been given an nth root, that is its nth root anytime it is needed.

Proof of Theorem 2.1. If b is an nth root of unity, then $h(bI)=\bar{b}I$. This is true because $h(bI)bI \geqslant 0$ and $[h(bI)bI]^n = h((bI)^n)(bI)^n = h(I) = I$ (by Lemma 2.1). Since every positive transformation has a unique positive nth root, it follows that $h(bI)bI = I$. Hence $h(bI)=\bar{b}I$.

Define a function $g: U(H) \to L(H)$ by

$$g(U) = \left[\,\overline{\det(U)}\,\right]^{1/n} h\!\left(U\left[\,\overline{\det(U)}\,\right]^{1/n}\right) \qquad \text{for all} \quad U \in \mathrm{U}(H).$$

(Note: the first paragraph of the proof shows that g is independent of the nth root that is chosen. See below for details.) This definition makes sense because $|\det(U)| = 1$ and

$$\det\!\left(U\left[\,\overline{\det(U)}\,\right]^{1/n}\right) = [\det(U)]\left[\,\overline{\det(U)}\,\right].$$

It follows immediately from the defining properties of h that $g(U)U \geqslant 0$. It is also true that $g(UV) = g(V)g(U)$. Here are the details:

$$\begin{aligned}
g(UV) &= \left[\,\overline{\det(UV)}\,\right]^{1/n} h\!\left(UV\left[\,\overline{\det(UV)}\,\right]\right)^{1/n} \\
&= \left\{\left[\,\overline{\det(U)}\,\right]^{1/n}\left[\,\overline{\det(V)}\,\right]^{1/n} e^{2i\pi m/n}\right\} \\
&\qquad \times h\!\left(UV\left[\,\overline{\det(U)}\,\right]^{1/n}\left[\,\overline{\det(V)}\,\right]^{1/n} e^{2i\pi m/n}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(for some integer } m\text{)} \\
&= \left[\,\overline{\det(V)}\,\right]^{1/n} h\!\left(V\left[\,\overline{\det(V)}\,\right]^{1/n}\right) \\
&\qquad \times \left[\,\overline{\det(U)}\,\right]^{1/n} h\!\left(U\left[\,\overline{\det(U)}\,\right]^{1/n}\right) e^{2i\pi m/n} h\!\left(e^{2i\pi m/n} I\right) \\
&= g(V)g(U).
\end{aligned}$$

Since $g(I) = h(I) \neq 0$, it follows that g satisfies the hypotheses of Lemma 2.3. Thus, $g(S) = S^*$ for all $S \in \mathrm{SU}(H)$. This shows that $h(S) = S^*$ for all $S \in \mathrm{SU}(H)$. ■

LEMMA 2.4. *Let $A(H) \subseteq \mathrm{L}(H)$ be a multiplicative semigroup which contains $U(H)$. Let $h: A(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in A(H)$. If $\Lambda = \{\lambda \in \mathbf{C} : \lambda I \in A(H)\}$,* *then there exists a multiplicative homomorphism* $\nu: \Lambda \to \mathbb{C}$ *such that* $h(\lambda I) = \nu(\lambda)I$ *for all* $\lambda \in \Lambda$.

Proof. If $h(I) = 0$, the lemma is proved. If $h(I) \neq 0$ then $h(I) = I$ (by Lemma 2.1), and thus the lemma is proved if $\dim H = 1$. Let $\dim H \geqslant 2$. If $J \in \mathrm{U}(H)$ is a symmetry, then $Jh(\lambda I) = h(J)h(\lambda I)$ (by Lemma 2.2) $= h(\lambda J) = h(\lambda I)h(J) = h(\lambda I)J$. This shows that $h(\lambda I)$ commutes with all symmetries in $\mathrm{L}(H)$. Thus, there exists a complex number $\nu(\lambda)$ such that $h(\lambda I) = \nu(\lambda)I$. Since $\nu(\lambda\mu)I = h(\lambda\mu I) = h(\lambda I)h(\mu I) = \nu(\lambda)\nu(\mu)I$, it follows that ν is multiplicative. ∎

Proof of Theorem 2.2. If $S \in \mathrm{U}(H)$ then

$$S\,\overline{[\det(S)]^{1/n}} \in \mathrm{SU}(H).$$

This implies that

$$h(S) = h\left(I\,[\det(S)]^{1/n}\right)\left(S\,\overline{[\det(S)]^{1/n}}\right)^* \qquad \text{(by Lemma 2.3)}$$

$$= h\left(I\,[\det(S)]^{1/n}\right)S^*\,[\det(S)]^{1/n}.$$

Since S is invertible, it follows that the transformation

$$h\left(I\,[\det(S)]^{1/n}\right)[\det(S)]^{1/n}$$

is independent of the nth root of $\det(S)$ that is chosen. Define τ by $\tau(\det(S))I = h(I\,[\det(S)]^{1/n})[\det(S)]^{1/n}$. The domain of τ is $\mathbb{T}$. The range of τ is contained in $\mathbb{R}^+$, because

$$h\left(I\,[\det(S)]^{1/n}\right)I\,[\det(S)]^{1/n} \geqslant 0,$$

$h(I\,[\det(S)]^{1/n})$ is a scalar multiple of I (by Lemma 2.4), and

$$I = h\left(I\,[\det(S)]^{1/n}\,\overline{[\det(S)]^{1/n}}\right)$$

$$= h\left(I\,[\det(S)]^{1/n}\right)h\left(I\,\overline{[\det(S)]^{1/n}}\right).$$

τ is well defined because $h(I\,[\det(S)]^{1/n})[\det(S)]^{1/n}$ is independent of the nth root of $\det(S)$ that is chosen. Thus, $h(S) = \tau(\det(S))S^*$. To complete the proof of this theorem, it must be demonstrated that τ is multiplicative. Well,

$$\begin{aligned}\tau(\det(S)\det(T))I &= \tau(\det(ST))I\\ &= h\left(I\left[\det(ST)\right]^{1/n}\right)\left[\det(ST)\right]^{1/n}\\ &= h\left(I\left[\det(T)\right]^{1/n}\left[\det(S)\right]^{1/n}e^{2i\pi m/n}\right)\\ &\quad\times\left[\det(S)\right]^{1/n}\left[\det(T)\right]^{1/n}e^{2i\pi m/n} \qquad \text{(for some integer } m)\\ &= \left\{h\left(I\left[\det(S)\right]^{1/n}e^{2i\pi m/n}\right)\left[\det(S)\right]^{1/n}e^{2i\pi m/n}\right\}\\ &\quad\times\left\{h\left(I\left[\det(T)\right]^{1/n}\right)\left[\det(T)\right]^{1/n}\right\}\\ &= \tau(\det(S))\tau(\det(T))\end{aligned}$$

as $(\det(S))^{1/n}e^{2i\pi m/n}$ is an nth root of $\det(S)$. ■

3. THE CURIOUS ROLE PLAYED BY SINGULAR TRANSFORMATIONS

LEMMA 3.1 [2, Corollary 3.1]. *Let $A(H) \subseteq L(H)$ be a multiplicative semigroup which contains $U(H)$, and let $h: A(H) \to L(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in A(H)$. If $h(I) \neq 0$, then each orthogonal (i.e. self-adjoint) projection P commuting with an element S of $A(H)$ also commutes with $h(S)$.*

Proof. The symmetry $J = 2P - I$ commutes with the same transformations as P. If h is applied to both sides of the equation $SJ = JS$, then it follows from Lemma 2.2 that $Jh(S) = h(S)J$. ■

LEMMA 3.2. *Let $A(H) \subseteq L(H)$ be a multiplicative semigroup which contains $U(H)$, and let $h: A(H) \to L(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in A(H)$. If P is an orthogonal projection, then $h(P) = P$ or $h(P) = 0$.*

Proof. If $h(I) = 0$ then $h(S) = 0$ for all $S \in A(H)$. So suppose $h(I) \neq 0$. Then $h(I) = I$ by Lemma 2.1. $h(0) = 0$, because $h(0) = h(0(-I)) = h(-I)h(0) = -h(0)$ (by Lemma 2.2). This shows that the result is true if

$\dim H = 1$. Let $\dim H \geqslant 2$, and let $O \neq P \neq I$ be an orthogonal projection. Then there exists a nontrivial subspace K of H such that $P = I \oplus O$ with respect to $K \oplus K^{\perp}$. It follows from Lemma 3.1 that $h(P) = P_1 \oplus P_2$ with respect to $K \oplus K^{\perp}$. If $J = I \oplus -I$ then $h(JP) = h(P)J$ (by Lemma 2.2) $= P_1 \oplus -P_2$ on the one hand, and $h(JP) = h(P) = P_1 \oplus P_2$ on the other hand. This shows that $h(P) = P_1 \oplus O$. Let $J \in L(K)$ be a symmetry. Then $J \oplus I$ is a symmetry which commutes with P. It follows from Lemma 2.2 that $J \oplus I$ commutes with $P_1 \oplus O$. This implies that P_1 commutes with all symmetries in $L(K)$. Thus, there exists a complex number b such that $P = bI$. Since $[h(P)]^2 = h(P^2) = h(P)$, it follows that $b^2 = b$. ∎

Corollary 3.1. *Let $A(H) \subseteq \mathrm{L}(H)$ be a multiplicative semigroup which contains* $\mathrm{GL}(H)$, *and let $h: A(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in A(H)$. If $h(R) = 0$ for some nonzero singular transformation $R \in A(H)$, then $h(T) = 0$ for all singular $T \in A(H)$.*

Proof. If $h(I) = 0$ or $\dim H = 1$, then the proof is complete. Let $\dim H \geqslant 2$ and $h(I) \neq 0$. Let $R \in A(H)$ be a nonzero singular transformation such that $h(R) = 0$. By the polar-decomposition theorem, there exists a unitary transformation U and a nonzero singular transformation $P \geqslant 0$ such that $R = UP$. This implies $h(P) = 0$, as $U^* \in A(H)$ and $h(P) = h(U^*UP) = h(R)h(U^*)$. Also, there exists an orthonormal basis for H such that $P = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m, 0, \ldots, 0)$ with respect to it, where $\lambda_i > 0$ for $i = 1, 2, \cdots, m$ $(m < n)$. If

$$P_1 = \mathrm{diag}\left(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_m^{-1}, 1, \ldots, 1\right)$$

with respect to the same basis, then $P_1 \in A(H)$, $Q = P_1 P$ is a nonzero orthogonal projection, and $h(Q) = 0$.

Let $S_0 \in A(H)$ be a singular transformation of maximal rank. That is, if $S \in A(H)$ and S is singular, then $\operatorname{rank} S \leqslant \operatorname{rank} S_0$. The method used to construct Q can be used to construct Q_0, an orthogonal projection which has the same rank as S_0. Since $A(H)$ contains all the unitary transformations, it follows that $A(H)$ contains all the orthogonal projections with the rank of S_0. Since $\operatorname{rank} Q \leqslant \operatorname{rank} S_0$, there exist orthogonal projections $Q_1, Q_2, \ldots, Q_k$ with the rank of S_0 for some $k < n$ such that $Q = Q_1 Q_2 \cdots Q_k$. Thus, $h(Q_k) \cdots h(Q_1) = h(Q) = 0$. It then follows from Lemma 3.2 that $h(Q_{i_0}) = 0$ for some $i_0 \in \{1, 2, \ldots, k\}$. If $T \in A(H)$ is any singular transformation, then there exists an orthogonal projection E of the same rank as S_0 such that $T = ET$. Also, there exists a unitary transformation W such that $E = WQ_{i_0}W^*$. Thus, $h(T) = 0$. ∎

REMARK 3.1. Let $A(H)$ be as in Corollary 3.1. The proof of Corollary 3.1 shows that if $A(H)$ contains a nonzero singular transformation, then $A(H)$ contains all orthogonal projections of rank 1.

Proof of Theorem 3.1. By Corollary 3.1, it may be assumed that $h(T) \neq 0$ for all nonzero singular T. Let S be an arbitrary element of $A(H)$. Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis for H. In the following, all matrices are written with respect to this basis. Let the matrix of S be (s_{ij}). It may be assumed without loss of generality that $|s_{ij}| \leqslant 1$ for $i, j = 1,2,\dots,n$. The reason is that if $N > 1$ then $I = h(I) = h(NI \cdot N^{-1}I) = h(N^{-1}I)h(NI)$. Thus, $h(N^{-1}I) = N^{-1}I$ implies $h(NI) = NI$. If $M > 1$ is large enough, then $|M^{-1}s_{ij}| \leqslant 1$ for $i, j = 1,2,\dots,n$. It then follows that $h(S) = h(M \cdot M^{-1}S) = (M^{-1}S)* MI = S*$.

Let b be an arbitrary entry in (s_{ij}). Then $b = s_{mp}$ for some $m, p \in \{1,2,\dots,n\}$. If P_k is the orthogonal projection onto the span of $\{e_k\}$, then $P_k \in A(H)$ (by Remark 3.1) and the matrix of $P_m SP_p$ has zeros everywhere except possibly in the mpth position; the entry in the mpth position is b. To prove $h(S) = S*$, it suffices to show that $t_{pm} = \bar{b}$, where $h(S) = (t_{ij})$. Let J be the transformation defined by $Je_p = be_m - (1-|b|^2)^{1/2}e_p$, $Je_m = \bar{b}e_p + (1-|b|^2)^{1/2}e_m$, $Je_k = e_k$ if $k \neq m$ or p. Since $|b| \leqslant 1$, it follows that J is a symmetry. Also, $P_m SP_p = P_m JP_p$. If h is applied to both sides of this equation, then it follows from Lemma 3.2 and Lemma 2.2 that $P_p h(S) p_m = P_p JP_m$. If each side of this equation is evaluated at e_m, then it follows that $t_{pm}e_p = \bar{b}e_p$. ∎

4. NONSINGULAR TRANSFORMATIONS

LEMMA 4.1. *Let $h: \mathrm{GL}(H) \to \mathrm{L}(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{GL}(H)$. If $h(I) \neq 0$, then there exists a multiplicative homomorphism $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ (ψ need not be continuous) such that either $h(P) = \psi(\det(P))P$ for all nonsingular $P \geqslant 0$ or $h(P) = \psi(\det(P))P^{-1}$ for all nonsingular $P \geqslant 0$.*

Proof. $h(T) \neq 0$ for all nonsingular T, as $I = h(TT^{-1})$. If $r \in \mathbb{R}^+$ then $h(rI) = \nu(r)I$, where $\nu: \mathbb{R}^+ \to \mathbb{R}^+$ is a homomorphism. This follows from Lemma 2.4 and the fact that $h(rI)rI \geqslant 0$. If $\dim H = 1$, then $\psi(P) = h(P)P$ works for all $P > 0$ or $\psi(P) = h(P)P^{-1}$ works for all $P > 0$. Let $\dim H \geqslant 2$. Let K be an $(n-1)$-dimensional subspace of H. In the following, all matrices are written with respect to $K \oplus K^{\perp}$. Let $W: K \to K^{\perp}$, let

$$S_W = \begin{pmatrix} I & 0 \\ W & -1 \end{pmatrix},$$

and let

$$h(S_W) = \begin{pmatrix} A & B \\ D & a \end{pmatrix}.$$

For $r \in \mathbb{R}^+$, let

$$Q_r = \begin{pmatrix} rI & 0 \\ 0 & r^{-1} \end{pmatrix}.$$

By Lemma 3.1, $h(Q_r) = \gamma(r) \oplus \delta(r)$ [here $\gamma(r)$ is an $(n-1)\times(n-1)$ matrix and $\delta(r)$ is a complex number]. If $J \in L(K)$ is a symmetry, then $J \oplus 1$ commutes with Q_r. This implies that $\gamma(r) \oplus \delta(r)$ commutes with $J \oplus 1$ (by Lemma 2.2). Thus, $\gamma(r) = \zeta(r) I$, where $\zeta(r) \in \mathbb{C}$. Since $h(Q_r)Q_r \geqslant 0$ and $h(Q_r)$ is invertible [with inverse $h(Q_{1/r})$], it follows that $\zeta(r) > 0$ and $\delta(r) > 0$. Since $Q_{rs} = Q_r Q_s$, it follows that ζ and δ are multiplicative homomorphisms on $\mathbb{R}^+$.

Let $0 < r < 1$ and let m be a nonnegative integer. Since $S_{r^{2m}W} = Q_{1/r^m} S_W Q_{r^m} = (Q_{1/r})^m S_W (Q_r)^m$, it follows that

$$h(S_{r^{2m}W}) = \begin{pmatrix} [\zeta(r)]^m I & 0 \\ 0 & [\delta(r)]^m \end{pmatrix} \begin{pmatrix} A & B \\ D & a \end{pmatrix} \begin{pmatrix} [\zeta(r)]^{-m} I & 0 \\ 0 & [\delta(r)]^{-m} \end{pmatrix}$$

$$= \begin{pmatrix} A & \left(\dfrac{\zeta(r)}{\delta(r)} \right)^m B \\ \left(\dfrac{\delta(r)}{\zeta(r)} \right)^m D & a \end{pmatrix}.$$

Let $b_r = \zeta(r)/\delta(r)$. Then $b_r > 0$. Since $h(S_{r^{2m}W})S_{r^{2m}W} \geqslant 0$, it follows that

$$\begin{pmatrix} A + b_r^m r^{2m} BW & -b_r^m B \\ b_r^{-m} D + ar^{2m} W & -a \end{pmatrix} \geqslant 0.$$

This shows that $a \leqslant 0$ and $-b_r^m B^* = b_r^{-m} D + ar^{2m} W$.

Can $b_r = 1$? If $b_r = 1$ then $-B^* = D + ar^{2m}W$. Since this is true for any integer $m \geqslant 0$, it follows that $a = 0$ and $-B^* = D$. Then

$$I = h(S_W^2) = [h(S_W)]^2 = \begin{pmatrix} * & * \\ * & -B^*B \end{pmatrix}$$

shows that $-B^*B = 1$. This is impossible. Thus, $b_r \neq 1$. It also follows that $a \neq 0$. That is, since $b_r \neq 1$, if $a = 0$ then $-b_r^m B^* = b_r^{-m} D + aW$ shows that $-b_r^{2m}B^* = D$. This implies that $B = 0$ and $D = 0$. This is impossible, as $h(S_w)$ is nonsingular.

Can $b_r > 1$? If $b_r > 1$, then $\|-B^*\| \leqslant b_r^{-2m}\|D\| + b_r^{-m}r^{2m}\|aW\|$ shows that $B^* = 0$, as the right side of the inequality can be made arbitrarily small. Thus, $b_r^{-m}D + ar^{2m}W = 0$. This shows that $-D = a(r^2 b_r)^m W$. This implies that $b_r = 1/r^2$ and $D = -aW$.

Can $b_r < 1$? If $b_r < 1$, then $\|-D\| \leqslant b_r^{2m}\|B^*\| + b_r^m r^{2m}\|aW\|$ shows that $D = 0$ as the right side of the inequality can be made arbitrarily small. Thus, $-b_r^m B^* = ar^{2m}W$. This shows that $-B^* = a(r^2 b_r^{-1})^m W$. This implies that $b_r = r^2$ and $B^* = -aW$.

It is impossible to have $b_{r_1} = r_1^2$ for some $0 < r_1 < 1$ and $b_{r_2} = 1/r_2^2$ for another $0 < r_2 < 1$, as this gives $D = 0$ and $B = 0$. (This implies $a = 0$.)

Case 1: $b_r = 1/r^2$ for all $0 < r < 1$. Then $h(Q_r) = \zeta(r)(I \oplus r^2)$ when $0 < r < 1$. Since $\zeta(1/r) = 1/\zeta(r)$ and $I = h(Q_r)h(Q_{1/r})$, it follows that $h(Q_{1/r}) = \zeta(1/r)(I \oplus 1/r^2)$ when $0 < r < 1$. Thus, $h(Q_r) = \zeta(r)(I \oplus r^2)$ for all $r > 0$. Let $V_r = I \oplus r$ with respect to $K \oplus K^{\perp}$. Then $V_r = r^{1/2} I \cdot Q_{1/r^{1/2}}$. Thus,

$$h(V_r) = \zeta(1/r^{1/2})(I \oplus 1/r)\nu(r^{1/2}) = \nu(r^{1/2})\zeta(1/r^{1/2})V_r^{-1}.$$

Let $\psi(r) = \nu(r^{1/2})\zeta(1/r^{1/2})$. Then $\psi: \mathbb{R}^+ \to \mathbb{R}^+$, ψ is a multiplicative homomorphism, and $h(V_r) = \psi(\det(V_r))V_r^{-1}$. This is independent of the choice of K. That is, if K_1 is any $(n-1)$-dimensional subspace of H and $T_r = I \oplus r$ with respect to $K_1 \oplus K_1^{\perp}$, then there exists a unitary transformation U such that $T_r = UV_rU^*$. Thus $h(T_r) = h(U^*)h(V_r)h(U) = \tau(\det(U^*))\tau(\det(U))\psi(\det(V_r))UV_r^{-1}U^*$ (by Theorem 2.2) $= \psi(\det(T_r))T_r^{-1}$. Since every nonsingular $P \geqslant 0$ is a product of transformations of the form $I \oplus r$, it follows that $h(P) = \psi(\det(P))P^{-1}$.

Case 2: $b_r = r^2$ for all $0 < r < 1$. Then $h(Q_r) = \zeta(r)(I \oplus 1/r^2)$ when $0 < r < 1$. The steps in case 1 can be repeated here if $I \oplus p$ and $I \oplus 1/p$ replace each other whenever $p = r$ or $p = r^2$. It then follows that $h(V_r) = \psi(\det(V_r))V_r$. Thus, $h(P) = \psi(\det(P))P$ for all nonsingular $P \geqslant 0$. ■

Proof of Theorem 4.1. If $\dim H = 1$, then $\sigma(T) = h(T)T$ works for all $T \in \mathrm{GL}(H)$ or $\sigma(T) = h(T)(T^*)^{-1}$ works for all $T \in \mathrm{GL}(H)$. Let $\dim H \geqslant 2$. If $S, T \in \mathrm{GL}(H)$, then there exist unique unitary transformations U_i and unique nonsingular transformations $P_i \geqslant 0$ for $i = 1,2,3$, such that $S = U_1 P_1$, $T = U_2 P_2$, and $ST = U_3 P_3$.

Case 1: $h(P) = \psi(\det(P))P^{-1}$ for all nonsingular $P \geqslant 0$ (Lemma 4.1). Then $h(S) = \psi(\det(P_1))\tau(\det(U_1))P_1^{-1}U_1^*$ (by Theorem 2.2) $= \sigma(\det(S))S^{-1}$, where $\sigma(\det(S)) = \psi(\det(P_1))\tau(\det(U_1))$. Clearly, $\sigma : \mathbb{C} - \{0\} \to \mathbb{R}^+$. To finish this case, it must be demonstrated that $\sigma(\det(S)\det(T)) = \sigma(\det(S))\sigma(\det(T))$. Well, $\sigma(\det(S))\sigma(\det(T)) = \psi(\det(P_1 P_2))\tau(\det(U_1 U_2))$ and $\sigma(\det(S)\det(T)) = \sigma(\det(ST)) = \psi(\det(P_3))\tau(\det(U_3))$. Since

$$\det(U_3)\det(P_3) = \det(U_1 U_2)\det(P_1 P_2),$$

$|\det(U_i)| = 1$, and $\det(P_i) > 0$ for $i = 1,2,3$, it follows that $\det(P_3) = \det(P_1 P_2)$ and $\det(U_3) = \det(U_1 U_2)$. Thus, $\sigma(\det(S)\det(T)) = \sigma(\det(S))\sigma(\det(T))$.

Case 2: $h(P) = \psi(\det(P))P$ for all nonsingular $P \geqslant 0$ (Lemma 4.1). Then $h(S) = \psi(\det(P_1))\tau(\det(U_1))P_1 U_1^*$ (by Theorem 2.2) $= \sigma(\det(S))S^*$. The analysis of σ done in case 1 also applies here. ■

5. AN APPLICATION

COROLLARY 5.1. *Let $\dim H \geqslant 2$. If b is a real number such that $b \neq \pm 1$, then there exist unitary transformations U_i and nonsingular transformations $P_i \geqslant 0$ for $i = 1,2,3$, such that $U_1 P_1 U_2 P_2 = U_3 P_3$ but $P_3^b U_3^* \neq P_2^b U_2^* P_1^b U_1^*$.*

Proof. Let R be a nonsingular transformation with (unique) polar decomposition UP, where U is a unitary transformation and $P \geqslant 0$ is nonsingular. Let b be a real number. Define the function h by $h(R) = P^b U^*$. Then $h : \mathrm{GL}(H) \to \mathrm{GL}(H)$. Suppose that whenever unitary transformations U_i and nonsingular transformations $P_i \geqslant 0$ satisfy the equation $U_1 P_1 U_2 P_2 = U_3 P_3$, they also satisfy the equation $P_3^b U_3^* = P_2^b U_2^* P_1^b U_1^*$. This implies that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{GL}(H)$. It then follows from Theorem 4.1 that $b = \pm 1$. ■

REMARK 5.1. This result is true for bounded linear transformations on a complex, infinite-dimensional Hilbert space. If H has dimension two, choose unitary transformations U_i and nonsingular transformations $P_i \geqslant 0$ that belong to $\mathrm{GL}(H)$ and that satisfy the equation $U_1 P_1 U_2 P_2 = U_3 P_3$ but $P_3^b U_3^* \neq P_2^b U_2^* P_1^b U_1^*$. Then consider $I \oplus U_i$ and $I \oplus P_i$.

REFERENCES

1 P. R. Halmos, *Finite-Dimensional Vector Spaces*, Springer, New York, 1974.
2 S. H. Hochwald, The uniqueness of the adjoint operation I, *Integral Equations and Operator Theory*, Vol. 9, 1986.

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